A COMPARISON OF HANSEN'S METHOD FOR PLANETARY PERTURBATIONS WITH ANDOYER'S VERSION OF IT

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Resumen: El método de Hansen para perturbaciones planetarias ha cobrado actualidad desde su reciente aplicación en algunos casos particulares de nuestro sistema planetario. La simplicidad con que provee las distintas perturbaciones conjuntamente con su particular estructura que permite un tratamiento mixto, analítico y numérico de las perturbaciones, lo han convertido en el método ideal para emprender la revisión de algunas teorías planetarias. Entre las aplicaciones más célebres se encuentra la llevada a cabo por Leveau el siglo pasado para desarrollar las perturbaciones del asteroide Vesta, teoría que aún mantiene su vigencia. El propósito de este trabajo es dar a conocer las características analíticas de una comparación que realizamos hace algunos años a pedido del Dr. G. M. Clemence, director a la sazón de la sección científica del Observatorio Naval de Washington entre la versión original de Hansen y la versión modificada del mismo realizada por Andoyer. De esta comparación ya hemos hablado en términos cualitativos en un número anterior de este Boletín. Dada la naturaleza del trabajo el lector deberá forzosamente referirse a las obras originales para seguir el curso de la comparación, ya que dada la extensión de ambas exposiciones se hace imposible reproducirlas aquí con todos sus detalles. En lo que sigue se indicará con H toda vez que nos refiramos al método tal cual fuera expuesto originalmente por Hansen y con A cuando se haga mención al procedimiento seguido por Andoyer.

In what follows I shall indicate Hansen's method with *H* and Andoyer's treatment with *A*. Numbers will be given to the different parts of the comparison in which analytical developments not given by Andoyer will be included when it will be necessary for the better understanding of the corresponding developed steps.

1 — In both methods their statement is made on the basis of an initial set of elliptics constants, to which numerical values are given. The developments are specially carried out for numerical computation of perturbations.

2 — A takes as reference plane a fixed plane P for the disturbed body M. This plane contains the origin of the coordinate system, and my be defined according to which sort of elements is chosen an absolute system of constants. This set may be an osculating set of elements at some time t^o, or mean elements appropriateli defined. It is clear that for another time t, the disturbed planet M will depart from plane P.
H adopts as reference plane, the plane of motion of the disturbed body, such that the radius-vector and the velocity vector must always stand on this plane. Thus the perturbations in this plane are divided in two classes; a) perturbations of the motion in the plane. b) perturbations due to the motion of the plane itself.

3 — A choses in the fixed plane P an "Initial ellipse". For this ellipse he defines the constants ε and $\tilde{\omega}$. The "reduced" radius-vector r, which is the cylin-

drical projection of the true radius-vector, and the mean longitude ν are put in a form such that they must satisfy the woll-known formulae for the elliptic motion:

$$r = b \left(1 + \frac{\varepsilon^2}{2} - \varepsilon \cos g - \frac{\varepsilon^2}{2} \cos 2g + \ldots \right)$$

$$\nu = \hat{\omega} + g + 2\varepsilon \sin g + \frac{\varepsilon^2}{4} \sin 2g + \ldots$$

where g is the disturbed mean anomaly and b is the disturbed semimajor axis. They shall define the transformations of the variables from which the new procedure will be outlined. Two other constants n and a are chosen for the "initial" ellipse, and such that:

$$n^2 a^3 = k^2$$

and also a constant g_0 such trat $g = nt + g_0 + \sigma$, together with

$$h_0 = n a^2 \cos \varphi \qquad \text{where} \qquad \varepsilon = \sin \varphi ;$$

h_0 corresponds to the areal constant in the two body problem. The last two constants are j_0 and θ_0 which define the position in space of plane P. In **H** the reference ellipse lies in the moving plane, and there will be valid the whole set of formulae for elliptic motion.

4 — Statement of the perturbations.

Writing the differential equitions for the disturbed motion in terms of the orthogonal components of the acceleration we have:

$$A \left\{ \begin{array}{ll} r^2 \dfrac{d\nu}{dt} = h & \dfrac{dh}{dt} = rH \\[2ex] \dfrac{d^2 r}{dt^2} - \dfrac{h^2}{r^3} = G & \dfrac{d^2 Z}{dt^2} = \kappa \end{array} \right.$$

These formulae can be obtained from principles stated in Tisserand's Traité de Mécanique Céleste T I, pages 461-474.
By a rather simple set of operations the system (A) is transformed into the following system:

$$* \left\{ \begin{array}{l} \dfrac{dg}{dt} = \dfrac{h}{b^2 \cos \varphi} \quad , \quad \dfrac{dh}{dt} = Q \quad , \quad \dfrac{d^2 Z}{dt^2} + \dfrac{k^2 Z}{r^3} = T \\[2ex] \dfrac{h}{b} = \rho \dfrac{k^2}{h} + \rho \cos \Psi \displaystyle\int F\, dt + \rho \sin \Psi \int G\, dt \end{array} \right.$$

where Ψ is the true longitude corresponding to the elements ε and ω F and G are quantities expressed in terms of other two Q and R, the coefficients of which depend on Ψ, r, k and h. The meaning of quantities Q, R and T will be explained later. Briefly speaking we can say here that their respective definitions depend on the sense in which the approximations are performed.

In **H** we have for the equations of motion in the moving plane of reference

$$\frac{d^2 r}{d t^2} - r \left(\frac{d v}{d t} \right) + \frac{\mu}{r^2} = R \quad ; \quad \frac{d}{d t} \left(r^2 \frac{d v}{d t} \right) = r S$$

The second of these equations is written in a rather different manner more appropriate for the manipulations

$$\frac{d}{d t} \left(\frac{1}{n} \right) = \frac{r}{\mu} S$$

Is is seen that the equations of motion in both methods differ from the structural point of view. In **A** the orthogonal components of acceleration are treated in an analytical form; in **H** they are obtained form the well-known set of equations defined by the method of arbitrary constants. This procedure facilitates the future computations of semi-analytical perturbations. Perturbations in latitude are also defined according these above mentioned precepts. We are now in conditions to define the equations for the perturbations. That means that in both methods the respective original set of equations of disturbed motion are appropriately transformed in such a way that only small quantities are to be computed.

For this purpose **A** puts:

$$g = nt + g_0 + \sigma \quad ; \quad b = \frac{1}{1 + \alpha} \quad ; \quad h = \frac{1}{1 + \beta}$$

and from the definitions it follows that: $z = a \zeta$. Where ζ is a small quantity of the first order. In fact $z = \sqrt{r^2 + z^2}$, where r is the true radius-vector. Developing the square root by the binomial theorem we get: $z = r \zeta$, and finally the elliptic formulae give $z = b \zeta$. In every case the expressión of is exact up to terms of second order in the defined small quantities. Then from the expression for b we have approximately $z = a \zeta$, the exactness of which is within small quantities of the first order. As in **A** we have to integrate an equation to obtain h/b, it must be defined a new intermediate unknown such that ω given by:

$$\frac{h}{b} = \frac{h_0}{a} \left(1 + \frac{\omega - \beta}{2} \right)$$

Let us see now **H** gives the values of small correcting perturbations. **H** defines the actual radius-vector y by $r_0 \ (1 + \nu)$.

Here r_0 is its value defined by elliptic motion and ν is the corresponding perturbation. The other correction to be computed is $n \, \delta \, t$. We define it through Kepler's equation, written in the form:

$$n \, (t + \delta \, t) = u - e \sin u ,$$

where u is eccentric anomaly. Perturbations in eccentricity and in the plane of motion are computed in **H** through small quantities ξ and η defined from the relations:

$$e \cos (\chi - \pi_0) = e_0 + \xi \, (1 - e^2_0)$$

$$e \sin (\chi - \pi_0) = \eta \, (1 - e^2_0)$$

where ξ and η are small quantities which will be associated with h in both derivatives. Introduction of ξ and η imply that $\frac{d\xi}{dt}(h\xi)$ and $\frac{d\eta}{dt}(h\eta)$ are expressed in terms of $\frac{d\xi}{dt}$ and $\frac{d\eta}{dt}$ these last derivatives being introduced by means of the method of the variations of arbitrary constants.

Perturbations in latitude in this method are given by the small quantity u defined from: $\mu = \frac{r_0}{a_0} S$

5 — The effective mechanism for the computations.

The computations in A depend on three auxiliary functions namely P_1, P_2 and P_3, when dealing with the perturbations in the plane used as reference. There is a fourth quantity Z for computing perturbations outside the plane. All these functions contain trems of different order which in general will not all used in the first approximation. In **H** the only quantity for calculating perturbations is named W. There is another determining function for the latitudes.

Let us see first the first order perturbations. For this purpose we must remain the form of the disturbing function used by A. It has the form:

$$U = V + \kappa^2 \left(\frac{1}{r} - \frac{1}{2} \frac{Z^2}{r^3} \right) + k^2 (1 + \kappa) \left(\frac{3}{8} \frac{Z^4}{r^5} + \dots \right)$$

in which V is the "proper" disturbed function. In dealing with two planets V will only depend on the reduced radius-vectors r and r'. V has then the form:

$$V = \frac{1}{\rho^*} - \frac{\rho^* \rho^{*\prime}}{\rho^{*3}} \cos H$$

A computes V by means of two terms:

a) $R' = \frac{1}{\triangle} - \frac{r}{r'^2} \cos H$; where $\triangle$ is expressed in terms of r and r'.

b) R_1 is a power series in the quantity $1/\triangle$ the exponents of which start with 3. The coefficients of the development depend on small quantities z, z' and of course on r, r' and lastly on ω^* which is a quantity of second order through z r', z' r and z z'. ω^* depends also on the angle J made by the fixed planes P and P' (P' fixed plane of reference for the planet M'). It is clear that for the disturbing planet M' it will be a corresponding auxiliary ellipse and its corresponding set of initial elliptic constants.

When the complementary part of the "proper" disturbing function V is transformed and developed in series we get:

$$\Delta R_2 = \frac{\omega^*}{r'^3} + (r\,r' \cos H + \omega^*) \left(\frac{3}{2} \frac{a'^2}{r'^5} b'^2 + \dots \right)$$

we see that in both parts of V appear the term

$$\rho^{*} \rho^{*\prime} \cos (\rho^{*} \rho^{*\prime})$$

To express this quantity in terms of r and r' we put:

$$\vec{\rho}^{*} = \vec{r} + \vec{Z} \quad , \quad \vec{\rho}^{*\prime} = \vec{r}' + \vec{Z}'$$

when the appropriate sense of these vectors is chosen. By multiplying both expressions, and then working out the second member, by writing the developed form of scalar products in both members, we obtain the desired expressión. We shall finally give the formula for ω^{*}:

$$\omega^{*} = r' Z \cos (\widehat{r' z}) + r Z' \cos (\widehat{r Z'}) + Z Z \cos (\widehat{Z Z'})$$

The angles in the second member as well as H (H the angle made by the reduced radius-vectors r and r') will be now expressed in terms of J and π and τ', J defined as usually, τ and τ' are the respective longitudes of the line of nodes of the planes P and P', both reckoned respectively on these plane in the same sense as ν and ν'.

We have already seen that W has a complementary part due to the fact that the mean motion n is defined from:

$$n^2 a^3 = k^2$$

while the value given by the observations is such that (A):

$$g = \nu^{*} t + g_0$$

The difference $n - \nu^{*}$ is of the order of the disturbing mass. To take into account this difference it is convenient to impose that $n^2 a^3$ satisfies the relations

$$n^2 a^3 = k^2 (1 + \kappa) = f (1 + m)$$

κ is a small constant, of the order of the masses. It must be understood that this last relation must also be fitted by variable values of n and a. This empirical change is permited if we modify the proper disturbing function V, by adding to it the term

$$\frac{k^2 \kappa}{\sqrt{r^2 + z^2}} \quad ;$$

and when this term is developed we really get the complementary part of W. We then can write for U:

$$U = f m' (R_1 + \Delta R_1) + \frac{k^2 \kappa}{\sqrt{r^2 + Z^2}}$$

We write for the successive approximations:

$$U = k^2 \left(\frac{1}{r} - \frac{1}{2} \frac{Z^2}{r^3} \right) + W$$

$$W = V + \kappa k^2 \left(\frac{1}{r} - \frac{1}{2} \frac{Z^2}{r^3} \right) + k^2 (1 + \kappa) \left(\frac{3}{8} \frac{Z^4}{r^5} + \dots \right)$$

On the other hand, we have from the elliptic motion:

$$G = -\frac{k^2}{r^2} \ , \ H = 0 \ , \ \kappa = -\frac{k^2 Z}{r^3}$$

In general for the successive approximations it is convenient to adopt the following set of definitions:

$$G = -\frac{k^2}{r^2} + R \ , \ r\,H = Q \ , \ \kappa = -\frac{k^2 Z}{r^3} + T$$

Then we have:

$$G = \frac{\partial U}{\partial r} \ , \ r\,H = \frac{\partial U}{\partial \nu} \ , \ \kappa = \frac{\partial U}{\partial Z}$$

and finally:

$$R = -\frac{3}{2}\,\frac{k^2 Z^2}{r^4} + \frac{\partial W}{\partial r} \ , \ Q = \frac{\partial W}{\partial \nu} \ , \ T = \frac{\partial W}{\partial Z}$$

It is then clear that W takes the place of a potential function. We are now ready to give a general account for the computatiin of the different perturbations. Is will be convenient here to start with H where we have for the perturbations in the plane of motion:

$$n_0\,\delta t = C_1 + \int \left[\nu^2 + \overline{W}_0 + \left(\frac{\partial \overline{W}_0}{\partial \lambda}\right) n_0\,\delta t + \left(\frac{\partial^2 \overline{W}_0}{\partial \lambda^2}\right) n_0^2\,\delta t^2 + \ldots \right] \frac{d\,t}{1-\nu^2}$$

$$\nu = C - \frac{1}{2} \int \left[\left(\frac{\partial \overline{W}}{\partial \lambda} \right) + \left(\frac{\partial^2 \overline{W}}{\partial \lambda^2} \right) n_0\,\delta t + \ldots \right] d\,l$$

For u we have also:

$$U = \int \left[\left(\frac{\partial \overline{R_0}}{d\,\tau} \right) \left(1 + \frac{d}{d\,t}\delta t \right) + \left(\frac{\overline{d^2 R_0}}{d\,t^2} \right) U_0\,\delta t + \ldots \right] d\,t$$

These formulae are rigorous. In each case it will only necessary in the first approximation to take into account terms of such order in the integrands.

One now should compute from H's theory the value of s and put its value together with the one for u for computing the integral:

$$\Gamma = \frac{h}{2\,\mu \cos^2 I_0} \int r\,S\,W\,d\,t$$

The term $\Gamma - \frac{1}{2}\,s\,\frac{d\,s}{d\,v}$ represents the correction to be added to v for obtaining L (mean longitude of the disturbed planet in a fixed plane) and it really represents the reduction from the plane of instantaneous motion to the fixed plane used to calculate the definitive motion. The quantity Γ results from a special transformation devised by Hansen to toke into account the fact that perturbations are initially computed with regard to a moving plane of reference.

Let us now turn our attention to the computation of perturbations in A. We write up to terms of second order:

$$\frac{h}{b} = \frac{h_0}{a}(1+\alpha)(1-\beta+\beta^2) = \frac{h_0}{a}\left(1 - \frac{\omega-\beta}{2}\right)$$

Then:

$$1-\beta+\beta^2+\alpha-\alpha\beta = \frac{\omega-\beta}{2}+1$$

and therefore:

$$\alpha(1-\beta) = \frac{\omega-\beta}{2}+\beta(1-\beta)$$

We next have always within terms of second order

$$\alpha = \frac{\omega-\beta}{2}(1+\beta)+\beta(1-\beta)(1+\beta)$$

and finally

$$\alpha = \frac{\omega+\beta}{2}+\beta\frac{(\omega-\beta)}{2}$$

On the other hand if γ is the unperturbed value of g it is evident that:

$$\frac{d\sigma}{dt} = \frac{dg}{dt} - \frac{d\tau}{dt} = \frac{h}{b^2\cos\varphi} - \frac{h_0}{a^2\cos\varphi} = \frac{h_0/(1+\beta)}{[a^2/(1+\alpha)^2]\cos\varphi} - \frac{h_0}{a^2\cos\varphi}$$

By developing this formula we get up to terms of second order:

$$\frac{d\sigma}{dt} = \frac{n\,a^2\cos\varphi\,(1-\beta+\beta^2)(1+2\alpha+\alpha^2)}{a^2\cos\varphi} - \frac{n\,a^2\cos\varphi}{a^2\cos\varphi} = 2\alpha-\beta+(\alpha-\beta)^2 =$$

$$= (2\alpha-\beta)+(\alpha-\beta)(\alpha-\beta)$$

But from the expression for α we have:

$$2\alpha-\beta = \omega+\beta\frac{(\omega-\beta)}{2}$$

$$\alpha-\beta = \frac{\omega-\beta}{2}+\beta\frac{(\omega-\beta)}{2}$$

Thus we shall have up to terms of the order quoted above:

$$\frac{1}{u}\frac{d\sigma}{dt} = \omega+\beta(\omega-\beta)+(\alpha-\beta)\frac{(\omega-\beta)}{2} =$$

$$= \omega+(\alpha+\beta)\frac{\omega-\beta}{2}n\ ;$$

$$\sigma = \sigma_0 + \int \left[\omega + (\alpha + \beta) \frac{\omega - \beta}{2} \right] n\, dt$$

The effective computation of perturbations follows from the second and third equations of group (°); we get

$$\beta = - \int \frac{h_0}{h^2} \frac{\partial W}{\partial \nu} dt$$

Without much difficulties it is finally shown that:

$$\beta = - \beta_1 - \int P_1 dt + \frac{\varepsilon}{2} \left(\beta_2 + \int P_2 dt \right)$$

$$\omega = - 3 \beta_1 - 3 \int P_1 dt + \left(\rho \cos \Psi + \frac{3 \varepsilon}{2} \right) (\beta_2 + \int P_2\, dt) +$$

$$+ \rho \sin \Psi \sec \varphi \, (\beta_3 + \int P_3\, dt)$$

P_1, P_2 and P_3 depend on quantities F and G and the derivative $\frac{\partial w}{d \nu}$, with coefficients which are functions of a, ε, h and h_0. In A the P's play the same role that the intermediate quantity, W in Hansen's theory, β_1, β_2, β_3 are constants of integration and of the order of the eccentricities. In A the derivative $\frac{\partial \omega}{\partial g}$ is needed also. It follows from

$$\frac{\partial \omega}{\partial g} = \frac{2}{(1 + \alpha)^2} \frac{d \alpha}{d t}$$

which in turn can be deduced from:

$$\frac{\partial}{\partial g} \left(\frac{h}{b} \right) = - \cos\varphi \frac{db}{dt}$$

It must bear in mind that h/b depend on the time through:

1) ρ, $\rho \cos \Psi$ $\rho \sin \Psi$

2) $\frac{k^2}{h}$, $\int F\, dt$, $\int G\, dt$

On the other hand it is clear that:

$$\frac{\partial \beta}{\partial g} = 0$$

In these calculations σ is of second orden with respect to denominators resulting from the integrations provided by the expression:

$$- \int\int P_1\, n\, dt^2$$

6 — The perturbations in latitude.

In both methods the first order computations do not depend on the perturbations in the respective planes if reference (plane of motion in **H**, fixed plane in **A**). In A one must solve the equation:

$$\frac{d^2 \zeta}{d t^2} + n^2 \frac{\zeta}{\rho_0{}^3} = \frac{Z}{a^2}$$

where:

$$Z = \frac{\partial W}{\partial \zeta} - n^2 a^2 \zeta^2 \left(\frac{a^2}{r^3} - \frac{1}{\rho^3{}_0} \right)$$

In the first step the second term of this last formula must not be taken into account. The solution can be carried out by the method of the variation of parameters. There are of course two constants if integration, for instance χ_1 and χ_2.

7 — The constants of integration

Before performing the integrations approximate values of the intermediate functions P_1, P_2, P_3 must be given:

$$P_1 = \frac{1}{n a^2} \frac{\partial W}{\partial g}$$

$$P_2 = \frac{2 \sec^2 \varphi}{n a^2} \left[\frac{\cos^2 \varphi - \rho^2}{\varepsilon} \frac{\partial W}{\partial g} + \rho \sin \Psi \sec \Psi \,.\, b \frac{\partial W}{\partial b} \right]$$

$$P_3 = \frac{2 \sec^2 \varphi}{n a^2} \left[\rho \sin \Psi \sec \varphi (\cos^2 \varphi + \rho) \frac{\partial W}{\partial g} - (\rho \cos \Psi + 2 \varepsilon) b \frac{\partial W}{\partial b} \right]$$

It must be remarked that the complete expressions of these functions contain small terms in ζ^2, which must be taken into account in higher approximations.

In **H** the determining function W and its derivative do not contain latitude terms.

In developing the complementary part W in **A** we take in the first approximation:

$$W = 2 \mu' n^2 a^2 R \overline{\sqrt{a a'}} + \kappa \frac{n^2 a^2}{\rho_0}$$

where

$$\mu' = \frac{1}{2} \frac{m' (1 + \kappa)}{1 + m} \sqrt{\frac{a}{a'}}$$

V is such that:

$$V = 2 \mu' n^2 a^2 (R \overline{\sqrt{a a'}} + \triangle R \overline{\sqrt{a a'}})$$

The quantity

$$2 \mu n^2 a^2 (R \overline{\sqrt{a a'}} + \triangle R \overline{\sqrt{a a'}})$$

is homogeneous of zero degree respect to lengths. If the longitudes are reckoned, from the common node of planes P and P′ the disturbing function in A has its final qualitative form:

$$R = \frac{1}{\sqrt{b\,b'}} \sum \cos [(S + q' + p_1 - p_2) g + (-S + q' + p'_1 + p'_2) g' +$$

$$+ (S + q') (\omega - \tau) + (-S + q') (\omega' - \tau')] \left(\frac{\varepsilon}{2}\right)^{p_1+p_2} \left(\frac{\varepsilon'}{2}\right)^{p'_1+p'_2}$$

$$N^{S+q'}_{p_1,p_2} \quad N^{-S+q'}_{p'_1,p'_2} \quad B^{q'}_{S}$$

Quantities $B_s^{q'}$ are linear and homogeneous functions of the laplacian coefficients b_n^p. The factors N depend on the so called Newcomb's operators. In this expression for R, s and q′ are arbitrary integers: p_1, p_2, p_1', pp_2' are non negative integers. Fronm this general description, it follows that the development of R is quite similar to the ones given by other methods in general planetary theory.

We can give here a brief discussion with regard the constants of integration. In A there are 13 constants, six of them are fixed *ab initio*: n (or a), ε, $\hat{\omega}$, g_0, j_0 and θ_0. If they are well determined the constants of integration, seven in full, must have small values. Seven constants appear in the integrations and then one of then can be taken at will. A puts:

$$\sigma_0 = 0$$

the six remaining are κ, β_1, β_2, β_3, χ_1, χ_2. These can be determined as follows.

σ and α can not have constant part; at the same time σ and ζ can not have terms in cos γ and sin γ respectively. In this case it is simply $\beta_1 = 0$; other ones are determined from the same quoted principles. Determination of κ.

As A states this determination follows from the necessity of putting into coincidence the values of n anl ν.

In order to do this A writes the conditions:

$$\beta = 0 \quad ; \quad \left(1 + \frac{d\sigma}{n\,dt}\right)^2 (1 + \alpha)^{-3} = 1 + \kappa$$

The first one corresponds to the fact that the complementary part of W does not depend on h. And the second one from the fact that it depends on r, and then on b. It is very easy to show from that:

$$\alpha = \kappa \quad ; \quad \omega = 2\kappa \quad ; \quad \sigma = (2\kappa + \kappa^2)\, n t$$

The influence of the complementary part of W over this calculations indicates that ω must be increased by 2π. To eliminate the constant part in ω and α we next have to take:

$$\beta_1 = 0 \quad ; \quad \mu = \mu' (-1 + 2D)\, b_0^{1/2}$$

where D is a differential operator and $b_0^{1/2}$ is a Laplacian coefficient. Remembering now that the change of the form of the "proper" disturbing function is such that the condition already quoted must be also satisfied by variable values of n and a. The form of the condition written by **A**

$$\left(1+\frac{d\sigma}{n\,dt}\right)^2(1+\alpha)^{-3}=1+\kappa$$

suggests the following proof to get it:

Let us take n_o as the value of the disturbed mean motion; as b is the disturbed value of the semi-major axis, and k is the Gaussian constant, we shall have:

$$n^2_o\,b^3 = k^2\,(1+\kappa)$$

according to the general method for computing perturbations we have in

$$n + n^o + \delta\,n + \delta^2 n + \ldots$$

n = constant defined by the equation: $n^2a^3 = k^2$, a is the undisturbed semi-major axis, n^o is a constant (of integration) and the $\delta^i n$ indicate different orders of perturbations. Besides we can write again according to the general method for computing perturbations:

$$g - l = (n + \nu^o)t + l^o + \delta\,l + \ldots$$

where l is the undisturbed mean anomaly (we have put here l instead of γ), l^o is a constant of integration and $\nu^o = n - \nu^*$. But in **A** we have

$$g - l = \sigma\text{; then}$$

$$\sigma = (n^o + \nu^o)\,t + l^o + \delta l + \delta^2 l + \ldots.$$

In this last expression δl, $\delta^2 l$, ... are sums of a secular part and a periodic part respectively. Then for instance:

$$\delta\,l = \delta\,l\,(\text{sec}) + \delta\,l\,(\text{per.})$$

$$\delta^2\,l = \delta^2\,l\,(\text{sec}) + \delta^2\,l\,(\text{per.})$$

But from the same theory:

$$\delta\,l\,(\text{sec}) = \mu' + n\,t\,b_o{}^{1/2}$$

μ' is of the order of the masses; f is a factor of this same order; b_o is a Laplacian coefficient. Besides:

$$\delta\,l\,(\text{per.}) = \mu'\,(\varepsilon_1\,\lambda - \varepsilon_2\,\lambda^{-1})$$

factors ε_1 and ε_2 are of the order of the eccentricities.

Then the factor of t in δ(sec) and the term δ(per) are small quantities of second order. We shall then obtain:

$$\sigma = [n^o + \nu^o + \delta\,l\,(\text{sec})]\,t$$

In this expression δl(sec) does not contain any factor t. Then from $g = n_o\,t + g_o$, we have

$$l + (n^o + \nu^o)\,t + l^o + \delta\,l + \ldots = (n + n^o + \delta\,n + \delta^2 n + \ldots)\,t + g_o$$

But

$$l = n\,t + g_o\ ;\ \text{and thus}$$

$$n^o + \delta\,n \cong (n^o + \nu^o + \delta\,l\,(\text{sec})$$

where $\delta l(\sec)$ does not contain any factor t. From this last equation and the expression for σ it follows that:

$$d\sigma \cong (n^o + \delta n)\, dt$$

Therefore we can write:

$$(n + n^o + \delta n + \ldots)^2 \frac{a^3}{(1+\alpha)^3} = k^2 (1 + \kappa)$$

On the other hand it is clear that:

$$\left(1 + \frac{d\sigma}{n\, dt}\right)^2 (1+\alpha)^{-3} n^2 a^3 = k^2 (1 + \kappa)$$

and the desired expression follows inmediatly.

8 — Numerical formulation of perturbations in A.

Only a brief account can be given here. A suggests the application of Cauchy's method for calculating the perturbations. Take P′ as reference plane, and express $\triangle$ in terms of arguments g and g′ and Ψ. Being g the mean anomaly of P, u′ the eccentric anomaly of P′, the true anomay of the first planet, we must compute the expressions $\left[\frac{a\,a'}{2}\right]$ in such a way that:

$$\left(\frac{a\,a'}{2}\right)^p = A^p \left[1 - \beta_0 \cos(g + n' + \Psi)\right]^{-p} \sum b^p{}_n\, e^{in\,(g - n' + \psi)}$$

β_0 is a small number; $A = \frac{2\,a\,a'}{\sin\chi}$; χ is an auxiliary angle such that: $0 \leqslant \chi \leqslant \frac{\pi}{2}$ and where p takes values 1/2, 3/2 ...; the $b^p{}_n$ are Laplacian coefficients where n takes integral values. Developing the first factor by using the binomial theorem we have:

$$\left(\frac{a\,a'}{2}\right)^p = \sum B^p{}_n\, e^{in(g - n')}\, e^{-q'g'}$$

The $B_n{}^p$'s are functions of the $b_n{}^p$, A^p and β and it contains the arguments g and Ψ. By giving fixed values to g it is possible to calculate the values of the series by means of harmonic analysis. It is clear that we must previously express the argument u′ in terms of g′ in which case we write:

$$e^{-in\,n'} = \sum \frac{n}{q'} J_{q'-n}(q'\,\varepsilon')$$

the $J_{q'-n}$ are Besselian functions of argument $q'-n$. To complete this transformations it will be necessary to put the different parts of the complementary function W in terms of the variable g. We have for W:

$$W = \mu' n^2 a^2 \left[\frac{\sqrt{a a'}}{\triangle} - \left(\frac{a}{a'} \right)^{3/2} \frac{\rho}{\rho'^2} \cos H \right]$$

In this form we shall be able to put all the quantities in appropriate form for the calculations. We shall also express the coefficients of the derivates of W in terms of periodic series with argument g. In order to facilitate the calculations it is convenient to put:

$$K_q = \frac{1}{2} \left[\frac{P_q}{P_1} + \frac{Q_q}{Q_1} \right]$$

Here P_q and Q_q are Besselian functions. The coefficientes K_{-2}, K_{-3} which appear in these developments are small.

According to the new form of the coefficientes $\rho \cos \Psi$, $i\rho \sin \Psi \sec \varphi$, $\dfrac{\cos^2 \varphi - \rho^2}{\varepsilon}$ and $i \rho \sin \Psi \sec \varphi \, (\cos^2 \varphi + \rho)$ the expressions for β, ω, and ζ can be written:

$$\beta = \int B_1 \, dt + \frac{\varepsilon}{4\Gamma} \int B_2 \, dt + \int B'_2 \, dt$$

$$\omega = 3 \int B_1 \, dt + F \int B'_2 \, dt + F' \int B_2 \, dt$$

$$\zeta = F_1 \int c' \, dt + F'_1 \int c \, dt$$

Of course B_1, B_2, B'_2, C and C′ are linear functions of $\dfrac{\partial W}{\partial g}$ and $\dfrac{\partial W}{\partial g}$, the coefficients of which are expressed in terms of ε, n, a and g. We have not included here the constants of integration. But we may say that: B_1 and β_1 are real quantities: B_2 and B'_2; C and C′; χ_1 and χ'_1 are respectively complex conjugate quantities; F and F′, G and G′ are also complex conjugate quantities respectively and where G and G′ are factors of $b \dfrac{\partial W}{\partial b}$ and $\dfrac{\partial W}{\partial g}$. As regards the perturbation α it is easily shown that it obeys to:

$$\frac{2}{n} \frac{d\alpha}{dt} = \frac{\partial F}{\partial g} \int B'_2 \, dt + \frac{\partial F'}{\partial g} \int B_2 \, dt$$

We must finally remember that the part $n^2 a^2$ of the disturbing function is taken into account by simply increasing ω by the amount 2π. The remaining constants of integrations are treated in quite similar manner.

9 — Perlurbations of second order.

We first deal with method A. Perturbations of this order in α and σ are computed by completing the corresponding definitions. We have for instance:

$$\alpha = \frac{\omega + \beta}{2} + \beta \frac{\omega - B}{2} \quad , \quad \sigma = \int \left[\omega + (\alpha + \beta) \frac{\omega - \beta}{2} \right] n\,dt$$

Then:

$$\delta \alpha = \frac{1}{2} (\delta \omega + \delta \beta \quad , \quad \delta \sigma = \int \delta \omega\, n\,dt$$

whose expressions give the increments to the perturbations just mentioned. In the new included terms higher order terms of three "proper" disturbing function V and its complementary part have to be added. This will imply considerations of derivatives of both functions. If, for instance, perturbations of second orden are required in the elements which depend on the perturbation ω we remember that:

$$\omega = 3 \int B_1\,dt + F \int B'_2\,dt + F' \int B_2\,dt$$

1) The perturbations due to the influence of changes in both factors F and F′ lead to compute directly:

$$\delta \omega = 2 \sigma \frac{d \alpha}{n\,dt}$$

2) We have next to consider lhe influences provided by changes in the integrands.

3) The third independent operation will consist of taking into account the influence of terms in ζ^2 through the intermediate functions P_1, P_2 and P_3.

4) The computation of first orden perturbations in the integrands.

When perturbations in latitude are calculated in a second approximation the second member of the corresponding differential equation must be completed. We shall then have:

$$\frac{d^2 \zeta}{dt^2} + n^2 \frac{\zeta}{\rho^3_o} = \frac{Z}{a^2} \quad , \quad Z = \frac{\partial W}{\partial \zeta} - n^2 a^2 \zeta \left(\frac{a^3}{r^3} - \frac{1}{\rho^3_o} \right)$$

Now Taylor's theorem gives for the last term of this second expresión:

$$3 n \zeta \left[\frac{\varepsilon \sin \Psi \sec \varphi}{\rho^4} \sigma - \frac{1}{\rho^3} \alpha \right]$$

on the perturbation ω we remember that:
in the first approximation. Needless to say that the coefficients of the development of W must also we put in terms of the argument g. The constants of integration are determined by taking numerical values pro-

vided by the first approximation and then calculating the values of their respective increments.

As regards the method devised by **H** the second order perturbations in t and r are taken into account by computing values of

$$\frac{1}{n} \frac{d}{dt} n \delta t$$

which are to be added to the previous $n \delta t$. For the perturbations in latitude we compute the correction. Both caculations imply the calculations by means of Taylor's theorem of the increments corresponding to functions T and U. The determination of the new values of the constants o fintegration is made by letting them keep their litteral form until the new approximation is performed.

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